Parametric resonant acceleration of particles by gravitational waves

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Abstract

We study the resonant interaction of charged particles with a gravitational wave propagating in the non-empty interstellar space in the presence of a uniform magnetic field. It is found that this interaction can be cast in the form of a parametric resonance problem which, besides the main resonance, allows for the existence of many secondary ones. Each of them is associated with a non-zero resonant width, depending on the amplitude of the wave and the energy density of the interstellar plasma. Numerical estimates of the particles' energisation and the ensuing damping of the wave are given.

1 Introduction

Despite the numerus efforts made up today to detect gravitational waves, there is no convincing evidence for their existence (Thorne 1987). This is due to the fact that not only their amplitude is very small (Smarr 1979), but it is highly possible that some kind of damping mechanism operates on them as they travel through space (Esposito 1971, Macedo and Nelson 1983, Papadopoulos and Esposito 1985). This damping may originate in the interaction of the gravitational wave with the interstellar matter (Macedo and Nelson 1990, Varvoglis and Papadopoulos 1992).

In a recent paper (Kleidis et al 1993, which hereafter is referred to as Paper I) the problem of the interaction of a charged particle with a gravitational wave, in the presence of a uniform magnetic field, has been modelled as a Hamiltonian dynamical system. The corresponding analysis was carried out for various directions of propagation of the wave with respect to the magnetic field. It was found that, in the oblique propagation, diffusive acceleration of the particle, due to secular energy transfer from the wave, could lead to a damping.

The most important results, however, came out from the parallel propagation case where the dynamical system is trapped at an exact resonance between the Larmor frequency of the particles and the frequency of the wave. In this case a phase lock situation appears (e.g. see Menyuk et al 1987), leading to an "infinite" acceleration of the particle and, consequently, to a non-trivial damping of the wave. The zero probability problem of the exact resonance was waived out in a more recent paper (Kleidis et al 1995, which hereafter is referred to as Paper II), by considering that the propagation of the gravitational wave takes place in a space filled with plasma, which results in a dispersion of the wave (Grishchuk and Polnarev 1980). In this case, resonant phenomena occur also in the quasiparallel case, i.e. propagation at a small angle with respect to the direction of the magnetic field, $\vartheta \leq 5^{\circ}$.

In the present paper we perform an elaborated study of the resonant interaction between charged particles and a linear polarized gravitational wave, which propagates in a non-empty space, parallel (or quasiparallel) to the direction of a uniform magnetic field $\mathbf{B} = B_0 \hat{e}_z$. The interstellar plasma is represented by a collisionless gas of particles, where by collisionless we simply mean that the mean time between succesive collisions of the particles is much larger than the period of the gravitational wave. As we show, in this case, the generalized coordinate x^1 obeys a Mathieu differential equation (Abramowitz and Stegun 1970). As long as the resonance condition given by Eq. (23) of Paper II is fullfiled, this equation corresponds to a dynamical system which is trapped at a parametric resonance, where any external action ammounts to a time variation of the frequency parameter (Landau and Lifshitz 1976). In this case, the system's equilibrium at rest $(x^1 = 0)$ is unstable. Any deviation from this state, however small, is sufficient to lead to a rapidly increasing displacement x^1 . Then, the corresponding generalized momentum π_1 satisfies a modified Mathieu equation, resulting in an exponential increase of the perpendicular energy, I_1 , of the particle. In this case, also, there exists a large number of secondary resonances of non-zero width (Bell 1957), at which the dynamical system may be trapped. Their exact location is given in terms of the energy parameter I_0 . The fact that the total measure of the resonant widths is non-zero increases the effectiveness of the interaction mechanism under study, both in the process of the acceleration of particles and in the damping of the wave.

2 Parametric resonance

We consider the non-linear interaction between a charged particle and a linear polarized gravitational wave propagating in a non-empty space. The interaction takes place in the presence of a uniform and static, in time, magnetic field, $\mathbf{B} = B_0 \hat{e}_z$, which does not interact with the gravitational wave. The motion of a charged particle in curved spacetime is given, in Hamiltonian formalism (Misner et al 1973) by

the differential equations

$$\frac{dx^{\mu}}{d\lambda} = \frac{\partial H}{\partial \pi_{\mu}} , \quad \frac{d\pi_{\mu}}{d\lambda} = -\frac{\partial H}{\partial x^{\mu}}$$
 (1)

where π_{μ} are the generalized momenta (corresponding to the coordinates x^{μ}) and the "super Hamiltonian", H, is given by the relation

$$H = \frac{1}{2}g^{\mu\nu} (\pi_{\mu} - eA_{\mu}) (\pi_{\nu} - eA_{\nu}) \equiv \frac{1}{2}$$
 (2)

(in a system of geometrical units where $\hbar = c = G = 1$). In Eq. (2) $g^{\mu\nu}$ denotes the components of the contravariant metric tensor, which are defined as

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$$

with $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ and $|h^{\mu\nu}| \ll 1$. A_{μ} is the vector potential, corresponding to the tensor of the electromagnetic field in a curved spacetime $F_{\mu\nu}$. The mass of the particle is taken equal to 1. For the specific form of the magnetic field we take

$$A_0 = A_1 = A_3 = 0$$
, $A_2 = -B_0 x^1$

According to Paper II the problem depends on four parameters: The normalized, dimensionless amplitude of the gravitational wave, α , the angle of propagation with respect to the direction of the magnetic field, ϑ , the dimensionless frequency, ν , (which is the ratio between the frequency of the wave ω and the Larmor frequency of the particles Ω) and the small parameter $\beta \sim \varrho/\omega^2$, which is a dimensionless constant depending on the energy density of the interstellar plasma ($\beta \ll 1$) (e.g. see Grishchuk and Polnarev 1980, Kleidis et al 1995).

Resonant phenomena become dominant, in particular, when the gravitational wave propagates parallel to the direction of the magnetic field, i.e. $\vartheta=0^{\circ}$. In this case, the non-zero components of the metric tensor are

$$g^{\mu\nu} = diag\left(1, -\frac{1}{1 - \alpha cosk_{\mu}x^{\mu}}, -\frac{1}{1 + \alpha cosk_{\mu}x^{\mu}}, -1\right)$$

and the super Hamiltonian, given by Eq. (2), is finally written in the form

$$H = \frac{1}{2}(2\nu - 1)\left(1 - \frac{\nu - 1}{\nu}\beta\right)I_0^2$$

$$- f(\nu, \beta)I_0I_3 - \frac{1}{2}\frac{\pi_1^2}{1 + \alpha sin\theta^3} - \frac{1}{2}\frac{(x^1)^2}{1 - \alpha sin\theta^3}$$
(3)

where, for the sake of convenience, we have set

$$f(\nu, \beta) = \nu \left(1 - \frac{2\nu - 1}{2\nu} \beta \right)$$

Since θ^0 is a cyclic coordinate, the corresponding generalized momentum I_0 is a constant of the motion and, according to Paper II, $I_0 \neq 0$. Now, the equation of motion for θ^3 is readily solved to give

$$\frac{d\theta^3}{d\lambda} = \frac{\partial H}{\partial I_3} \quad \Rightarrow \quad \theta^3 = \frac{\pi}{2} - f(\nu, \beta)I_0\lambda \tag{4}$$

where $\pi/2$ is the initial phase. Since $\alpha \ll 1$, in what follows we keep terms up to first order in α . Using the approximation

$$\frac{1}{1 \pm \alpha sin\theta^3} \approx 1 \mp \alpha sin\theta^3$$

the geodesic differential equation of motion for x^1 is written in the form

$$\frac{d^2x^1}{d\lambda^2} + \alpha f(\nu, \beta)I_0 \sin\left[f(\nu, \beta)I_0\lambda\right] \frac{dx^1}{d\lambda} + x^1 = 0 \tag{5}$$

where we have also used the equation of motion for π_1 and the fact that

$$\pi_1 = \frac{dx^1}{d\lambda} + eA_1$$

in which, for the specific form of the magnetic field, $A_1 = 0$.

We consider a transformation of the independent variable, λ , of the form $s = s(\lambda)$, where $s(\lambda)$ is chosen as to satisfy the differential equation

$$\frac{d^2s}{d\lambda^2} + \alpha f(\nu, \beta) I_0 sin \left[f(\nu, \beta) I_0 \lambda \right] \frac{ds}{d\lambda} = 0$$
 (6)

The condition (6) guarantees that the resulting, in terms of s, equation of motion for x^1 will continue to represent the differential equation of a geodesic (Papapetrou 1974). Eq. (6) may be solved in the first order to α , to give

$$s(\lambda) \simeq \Omega\left(\lambda + \frac{\alpha}{f(\nu, \beta)I_0}sin[f(\nu, \beta)I_0\lambda]\right)$$
 (7)

where the Larmor frequency, Ω , appears as an integration constant. Now, in terms of $s(\lambda)$, Eq. (5) may be written in the form

$$\frac{d^2x^1}{ds^2} + \Omega^2 (1 - 2\alpha \cos[f(\nu, \beta)I_0\Omega s]) x^1 = 0$$
 (8)

where we have also used the fact that $\alpha \ll 1$. This is a Mathieu equation (Abramowitz and Stegun 1970) which is closely related to the problem of *parametric resonance* in dynamical systems (Landau and Lifshitz 1976). According to it, the external action (in our case the gravitational wave) amounts only to a periodic time variation of the

frequency parameter of the unperturbed system (the Larmor frequency, Ω). Indeed, in this case, the generalized frequency

$$\sigma^2(s) = \Omega^2[1 - 2\alpha \cos(\gamma s)] \tag{9}$$

differs only slightly from the constant Ω and is a simple periodic function. In Eq. (9) $\gamma = f(\nu, \beta)I_0\Omega$ denotes the frequency of this periodic function. It can be proved (Landau and Lifshitz 1976) that the main parametric resonance in a dynamical system appears when $\gamma = 2\Omega$.

It has been shown (Kleidis et al 1995) that the system gravitational wave + charged particle is trapped in a resonance when the resonant condition, given by Eq. (23) of Paper II, is satisfied, that is when

$$f(\nu,\beta)I_0 = 2\tag{10}$$

which gives $\gamma = 2\Omega$. Therefore, once the condition (10) is satisfied, our dynamical system is actually trapped in a main parametric resonance. In general, there is a region of instability of non-zero measure, the so called resonant width (Landau and Lifshitz 1976), around the exact value $\gamma = 2\Omega$, inside which the parametric resonance mechanism operates, which is

$$2\Omega - \frac{1}{2}\Delta\epsilon \le \gamma \le 2\Omega + \frac{1}{2}\Delta\epsilon \tag{11}$$

The range (Bell 1957) of the resonant width is given, in our case, by $\Delta \epsilon = 2\alpha\Omega$.

Apart from the case where γ is close to 2Ω , parametric resonance also occurs when the frequency of the parameter $\sigma(s)$ is close to any value $2\Omega/n$, where n is a natural number (Landau and Lifshitz 1976). Therefore, except from the primary resonance, there is also a large number of secondary ones at which the dynamical system under consideration may be trapped. Each of them also has a non-zero resonant width, the exact form of which, in terms of Ω , is given by (Bell 1957)

$$\Delta \epsilon^{(n)} = (\frac{n}{2})^{2n-3} \alpha^n \frac{1}{[(n-1)!]^2} \Omega$$
 (12)

that is, it decreases rapidly with increasing n, as α^n .

In our case it is more convenient to express both the possition of each of the secondary parametric resonances and the corresponding resonant widths in terms of the constant of the motion I_0 , which corresponds to a measure of the total energy of the dynamical system. Since, according to Paper II, the exact resonance is achieved for

$$I_0 = I_0^* = \frac{2}{f(\nu, \beta)}$$

the dynamical system will be trapped at a secondary one, of order n, when

$$I_0 = \frac{I_0^{\star}}{n} \tag{13}$$

For each of these secondary resonances the corresponding resonant width is given by

$$\Delta I_0^{(n)} = \frac{\Delta \epsilon^{(n)}}{f(\nu, \beta)} \frac{1}{\Omega}$$

or else

$$\Delta I_0^{(n)} = \frac{1}{f} \left(\frac{n}{2}\right)^{2n-3} \alpha^n \frac{1}{[(n-1)!]^2}$$
 (14)

where $f = f(\nu, \beta)$. Using Stirling's formula, Eq. (14) is simplified to

$$\Delta I_0^{(n)} = \frac{4}{\pi f} \frac{1}{n^2} \left(\frac{e}{2}\sqrt{\alpha}\right)^{2n} \tag{15}$$

If, furthermore, we set

$$\chi = \frac{e^2}{4}\alpha < 1$$

the resonant width of a parametric resonance of order n is finally written in the form

$$\Delta I_0^{(n)} = \frac{4}{\pi f} \frac{\chi^n}{n^2} \tag{16}$$

In this case, we may, also, calculate the *total measure* of the resonant widths, in terms of I_0 , which corresponds to the probability of the dynamical system to be trapped in a resonance. The total measure of the resonant widths is written in the form

$$\sum \Delta I_0 = \frac{4}{\pi f} \sum_{n=1}^{\infty} \frac{\chi^n}{n^2} \tag{17}$$

As long as $\chi < 1$, the series on the r.h.s of Eq. (17) satisfies D' Alembert's criterion (e.g. see Gradshteyn and Ryzhik 1965) and therefore converges absolutely and uniformly. In this case the total measure of the resonant widths can be written as

$$\sum \Delta I_0 = \frac{4}{\pi f} \chi_3 F_2(1, 1, 1, ; 2, 2; \chi)$$
 (18)

where we have taken into account the properties and the series expansion of the generalized hypergeometric series ${}_3F_2(1, 1, 1; 2, 2; \chi)$ (e.g. see Erdelyi et al 1953). If we keep only terms of first order with respect to α and β , Eq. (18) is written in the form

$$\sum \Delta I_0 = \frac{e^2}{\pi \nu} \alpha + O(\alpha \beta) + O(\alpha^2)$$
 (19)

where e is the basis of the natural logarithms.

3 Numerical results

The most interesting case of interaction between a gravitational wave propagating in a non-empty space and a charged particle is the quasiparallel case, in which the propagation takes place at a small angle with respect to the direction of the magnetic field.

Since, in this case, the phase velocity of the gravitational wave is greater than the velocity of light (Grishchuk and Polnarev 1980), it is probable that its projection along the z-axis, $v_z = v_{ph} cos \vartheta$, to be exactly equal to c. This leads (see Paper II) to a combined interaction (both chaotic and resonant) between the particle and the wave, resulting in an overall decrease in the parallel momentum I_3 (which corresponds to an overall increase of the perpendicular one I_1) as a function of s (see Fig. 1a).

Fig. 1a and b: Plots of I_3 versus s (the affine parameter) for a numerically integrated trajectory with $\vartheta = 0.5^{\circ}$, $\nu = 52.01$, $\beta = 10^{-5}$ and $\alpha = 0.002$. Notice the overall decrease in I_3 when (a) $I_0 = I_0^{\star} = 0.038$, in contrast to the purely chaotic behaviour when (b) $I_0 = 0.0385$ which lies outside the corresponding theoretical resonant width $\Delta I_0 = 7 \cdot 10^{-5}$.

Since our problem corresponds to a parametric resonance, except from the primary resonant condition, for $I_0 = I_0^*$, there is also a large number of values of the constant of motion I_0 , $I_0 = I_0^*/n$ (n = 1, 2, 3, ...), leading to secondary resonances, each of which has a non-zero resonant width. One expects that the above mentioned theoretical results in the quasiparallel propagation case should continue to hold around each of these secondary resonances, provided that the values of I_0 lie within the corresponding resonant width, $\Delta I_0^{(n)}$ (in connection to this see Fig. 1b).

To check for the validity of these theoretical estimates, we integrate numerically the equations of motion of a charged particle in a gravitational wave for several values of the constant of motion I_0 , when $I_0^* = 0.038$. Some of the corresponding results are presented in Figs. 2 and 3. According to these, when $I_0 = I_0^*/n$, we do observe the expected decrease in I_3 , which corresponds to the exponentially increasing brantch of I_1 , and implies that our dynamical system is trapped in a parametric resonance.

Fig. 2a and b: Plots of I_3 versus s for a numerically integrated trajectory with $\vartheta = 0.5^{\circ}$, $\nu = 52.01$, $\beta = 10^{-5}$ and $\alpha = 0.002$ when (a) $I_0 = I_0^{\star}/2$ and (b) $I_0 = I_0^{\star}/5$

The fact that the interaction of a charged particle with a gravitational wave in the parallel and/or the quasiparallel case corresponds to a parametric resonance, may impose noteworthy implications to the problem of the interaction between charged particles and a gravitational wave, increasing both the efficiency and the probability to achieve this particular interaction mechanism.

Fig. 3a and b: Similar to Figs. 2, except that (a) $I_0 = I_0^{\star}/9$ and (b) $I_0 = I_0^{\star}/12$

As a gravitational wave propagates in the non-empty interstellar or intergalactic

space, it encounters several clouds with various characteristic temperatures and densities, while at the same time, it intersects the cosmic magnetic fields at various angles (Hillas 1984). In each case corresponds a different set of values of β , ν and ϑ . One expects that, at least for some of these sets, one of the resonant conditions is satisfied (since the corresponding resonant widths are non-zero), leading to a resonant interaction between a charged particle and the gravitational wave. According to Paper II, the result of this interaction is the acceleration of the particle at very high energies. Since the total energy I_0 is constant, the dynamical system (gravitational wave and charged particle) is *isolated*. Therefore any energisation of the particle corresponds to a damping of the wave.

It is worth to note that the decrease of I_3 in Figs. 2 and 3 is more pronounced for large values of n. This is in contrast to what one may have expected, since the higher order resonances are considered to be weaker than the lower order ones. However this result is only phenomenological and of no physical importance, since it has to do only with the slope of the straight line $I_3 - I_1$. In complete correspondence with Paper II we may find, from the condition $H_0 \simeq 1/2$, that, in momentum space, the particles move along the straight line

$$I_3 = -\frac{1}{2} \left(\frac{I_0^{\star}}{I_0} \right) I_1 + \left[\frac{2\nu - 1}{\nu^2} (1 + \beta) \frac{I_0 f(\nu, \beta)}{2} - \frac{1}{4} \left(\frac{I_0^{\star}}{I_0} \right) \right]$$
(20)

and since, at every n^{th} -order parametric resonance we have $I_0 = I_0^*/n$, Eq. (20) can be written in the form

$$I_3 = -\frac{n}{2}I_1 + \left[\frac{1}{n}\frac{2\nu - 1}{\nu^2}(1+\beta) - \frac{n}{4}\right]$$
 (21)

We note that for n=1 (the primary resonance) Eq. (21) is reduced to the corresponding one of Paper II [Eq. (28)]. In the present case, as $n \to \infty$, the straight line $I_3(I_1)$ tends to become parallel to the I_3 -axis. This simply means that a small variation in I_1 leads to a large variation of I_3 , which is exactly the case shown in Figs. 2 and 3.

However, there is another numerical result which may be of physical importance. It is related to the fact that, for several of the secondary resonances, resonant phenomena are observed even for values of I_0 lying outside of the corresponding, theoretically estimated, resonant widths [Eq. (14)]. It should be noted that this purely numerical result is not observed in the case of the primary resonance, $I_0 = I_0^{\star}$. Some of these results are shown in Figs. 4.

In Fig. 4a we present a plot of I_3 versus s for a numerically integrated trajectory with $I_0 = \frac{I_0^*}{2} + 10^{-4}$. We see that an overall decrease in I_3 occurs although, in this case, $\Delta I_0^{(2)} \simeq 10^{-7}$. The same is also true for the case presented in Fig. 4b, where $I_0 = \frac{I_0^*}{9} + 10^{-6}$, far away from the corresponding resonant width, which, in this case is $\Delta I_0^{(9)} \simeq 10^{-27}$. Therefore, it seems that the numerically estimated resonant width, $\delta I_0^{(n)}$, around each of the higher-order parametric resonances are many orders of magnitude larger than the corresponding theoretical one.

Fig. 4a and b: Plots of I_3 versus s for a numerically integrated trajectory corresponding to a problem of parametric resonance which is (a) similar to the one of Fig. 2a, but with $I_0 = 0.0191 = \frac{I_0^*}{2} + \delta I_0^{(2)}$ and (b) similar to the one of Fig. 3a, but with $I_0 = 0.004221 = \frac{I_0^*}{9} + \delta I_0^{(9)}$. In any of these cases $\delta I_0^{(n)} \gg \Delta I_0^{(n)}$, but also $\delta I_0^{(n)} \ll \sum \Delta I_0$.

This behaviour results probably from the fact that, because of the small value of I_0^* , the secondary resonances are closely spaced. Therefore, even a small diffusion process due to the chaotic motion, which is always present in this case, could probably bring the value of I_0 into a resonance, even if, initially, it was outside of it. Indeed, as we may see from Figs. 4, the resonant behaviour becomes evident only after a considerable time interval. This time interval should correspond to the correlation time of the diffusive process (Farina et al 1993). Therefore, our system may be a lot more efficient, with respect to the resonant interaction procedure, than the corresponding theoretical one. Notice also, that these phenomena are not present in the case of the primary resonance, since the corresponding value of I_0 lies far away from every of the secondary ones and hence, it possesses a well defined resonant width. For this reason, in this case, theoretical and numerical results are similar.

4 Exact solutions

In Paper II it has been shown that, when the gravitational wave propagates exactly parallel to the direction of the magnetic field (the z-axis), we may solve the equations of motion of the charged particle. Their solution reveals a *phase-lock* situation (Menyuk et al 1987, Karimabadi et al 1990) resulting in an "infinite" acceleration of the particle along the x-axis. However, there, the solution was obtained through an averaging technique (e.g. see Lichtenberg and Lieberman 1983). Since the problem is identified as that of a parametric resonance, in which the equation of motion along the x-axis corresponds to a Mathieu equation (Gradshteyn and Ryzhik 1965, Abramowitz and Stegun 1970), we may find exact solutions without resorting to any averaging technique.

In the parallel propagation case, $\vartheta = 0^{\circ}$, the dynamical system is trapped in a resonance when the condition (10) is satisfied. In this case the equation of motion (8), for the generalized coordinate x^{1} , is written in the form

$$\frac{d^2x^1}{ds^2} + \Omega^2[1 - 2\alpha\cos(2\Omega s)] x^1 = 0$$
 (22)

or else

$$\frac{d^2x^1}{d\xi^2} + \Omega^2[1 - 2\alpha\cos(2\xi)] x^1 = 0$$
 (23)

where we have set $\xi = \Omega s$. This is a Mathieu equation with a = 1 and $q = \alpha$ (Gradshteyn and Ryzhik 1965, Abramowitz and Stegun 1970). Its solution is therefore

given in terms of the Mathieu functions ce_1r and se_1 , associated with even and odd periodic solutions respectively. Each one of these solutions has a period of π or 2π . In the present paper we consider periodic solutions of period 2π . It can be shown that, for a given point (a,q) in the parameter space, there can be at most one periodic solution of period 2π (Abramowitz and Stegun 1970) and therefore our solution is unique. The general solution of Eq. (23) is written in the form

$$x^{1}(\xi) = A \operatorname{ce}_{1}(\xi, \alpha) + B \operatorname{se}_{1}(\xi, \alpha)$$
(24)

where A, B are constants.

For $\alpha \ll 1$ the Mathieu functions ce_1 and se_1 may be expanded in power series of α . Considering that A = C = B, in the linear approximation to α (e.g. see Abramowitz and Stegun 1970, Eq. 20.2.27, p. 725), Eq. (24) becomes

$$x^{1} = \sqrt{2}C\sin\left(\frac{\pi}{4} + \xi\right) - \alpha\frac{\sqrt{2}}{8}C\sin\left(\frac{\pi}{4} + 3\xi\right)$$
 (25)

In this case, using the normalization conditions of the Mathieu functions (e.g. see Gradshteyn and Ryzhik 1965, Eqs. 6.911.3, 6.911.5. and 6.911.7), we may determine the exact value of the constant C. The mean square displacement of x^1 is defined as

$$\langle (x^1)^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left[x^1(\xi) \right]^2 d\xi = C^2$$
 (26)

so that

$$C = \sqrt{\langle (x^1)^2 \rangle}$$
 (27)

Now, Eq. (25) is written in the form

$$x^{1}(s) = \sqrt{2}\sqrt{\langle (x^{1})^{2} \rangle} sin\left(\frac{\pi}{4} + \Omega s\right) - \alpha \frac{\sqrt{2}}{8}\sqrt{\langle (x^{1})^{2} \rangle} sin\left(\frac{\pi}{4} + 3\Omega s\right)$$
 (28)

and therefore, for $\alpha = 0$, initially (s = 0) we have

$$x_0^1 (s=0) = \sqrt{\langle (x^1)^2 \rangle}$$
 (29)

corresponding to a sort of a Brownian motion (a not unexpected result).

However, the most important results come out from the solution of the equation of motion for the generalized momentum π_1

$$\frac{d\pi_1}{d\lambda} = \frac{x^1}{1 - \alpha sin\theta^3} \tag{30}$$

To solve this equation we follow the same procedure as in the case of the generalized coordinate x^1 . We then find that the perpendicular momentum π_1 obeys a modified Mathieu equation (e.g. see Abramowitz and Stegun 1970), of the form

$$\frac{d^2\pi_1}{ds^2} - \Omega^2 \left[1 - 2\alpha \cosh(2\Omega s) \right] \pi_1 = 0 \tag{31}$$

Its general solution is given in terms of the *radial* Mathieu functions Ce_1 and Se_1 (Gradshteyn and Ryzhik 1965, Abramowitz and Stegun 1970)

$$\pi_1(s) = A Ce_1(\Omega s, \alpha) + B Se_1(\Omega s, \alpha) = A ce_1(i\Omega s, \alpha) - iB se_1(i\Omega s, \alpha)$$

which, in the first approximation to α , is written in the form

$$\pi_1 = A \left[\cosh(\Omega s) - \frac{\alpha}{8} \cosh(3\Omega s) \right] + B \left[\sinh(\Omega s) - \frac{\alpha}{8} \sinh(3\Omega s) \right]$$
(32)

Compatibility of these results with those of Paper II requires that, in this case, we have to set B=0 thus obtaining

$$\pi_1 = A \left[\cosh(\Omega s) - \frac{\alpha}{8} \cosh(3\Omega s) \right] \tag{33}$$

From Eq. (33) it is evident that there is an exponential increase in the perpendicular momentum of the particles, which, in the zeroth order approximation to α , corresponds to the exponential branch of Eq. (27) of Paper II. We also note that, in the exact solution case, an additional, secularly increasing term, $\sim \alpha \cosh(3\Omega\lambda)$, arises. The corresponding perpendicular energy, $I_1(\lambda)$, in the first order approximation to α , is given by

$$I_1 \sim \frac{1}{2}cos^2\left(\lambda + \frac{\alpha}{2}sin2\lambda\right) cosh(\alpha\lambda) \left[1 + \frac{\alpha}{2}cos(3\lambda)cosh(3\alpha\lambda)\right] + \dots$$
 (34)

where we have used Eq. (7). Eq. (34) is in complete agreement to the results of Papers I and II. We see that the exponential increase in the perpendicular energy is, furthermore, modulated by periodic functions of time, which were smoothed out by the averaging technique in the previous Papers. It is important to point out that the phase-lock situation breaks down when I_1 becomes large enough, since then the perturbation term of the Hamiltonian, H_1 , becomes comparable to H_0 and our approximation, in terms of the small parameter α , is not valid any more.

Based on these results we may give an estimate of the absorption power per unit area of the gravitational wave, as a function of the proper distance from its source to Earth, for typical parameters of the interstellar gas.

In physical units, the energy density gained by the charged particles due to their resonant interaction with an incident gravitational wave, within a proper time interval $\Delta \lambda$, is

$$\Delta E_{gained} = \Delta I_1 m_P c^2 \left(\frac{n_{act}}{n_{tot}}\right) n_{tot} \tag{35}$$

where m_P is the proton's mass, (n_{act}/n_{tot}) is the ratio of particles which interact resonantly with the gravitational wave, n_{tot} is the particles' number density of the interstellar matter and ΔI_1 is the energy change within the interval $\Delta \lambda$, measured in dimensionless units.

Using Eq. (29) of Paper II, also translated in physical units, for $\nu = 1$ we obtain

$$\Delta E_{gained} = \frac{1}{2} \Delta I_1 m_P c^2 \sqrt{\frac{m_P c^2}{2\pi k_B T}} \beta n_{tot}$$
 (36)

Accordingly, we consider that the interstellar space is filled with a non-relativistic perfect proton gas (Polnarev 1972), for which

$$\beta = 16\pi G \frac{\rho}{\omega^2} \frac{k_B T}{m_P c^2} \quad , \quad \rho = m_P n_{tot} \tag{37}$$

We take $n_{tot} = 1particle/cm^3$ and T = 10K as typical mean values for the interstellar space. Then, we find that the total energy density gained by the charged particles due to their resonant interaction with the gravitational wave, within the proper time interval $\Delta \lambda$, is

$$\Delta E_{gained} = 1.61 \times 10^{-39} \frac{\Delta I_1}{\omega^2} \quad (\frac{erg}{cm^3}) \tag{38}$$

Since, on the other hand, the system wave + particles is *isolated*, Eq. (38) corresponds also to the total energy density lost from the gravitational wave, due to the interaction mechanism under consideration.

The corresponding gravitational energy flux (power per unit area) lost in the same proper time interval is

$$\Delta F_{lost} = 1.21 \times 10^{-29} \frac{\Delta I_1}{\omega^2} \quad (\frac{erg}{cm^2 sec}) \tag{39}$$

Using Eq. (27) of Paper II for ΔI_1 , we integrate Eq. (39) over λ , to obtain the total loss of gravitational energy flux during the whole proper time interval $0 \le \lambda \le \lambda_E$, at which the wave travels from its source to Earth. Taking into account that for every realistic proper time interval λ_E we have $\alpha \lambda_E \ll 1$, we obtain

$$F_{lost} = 6.1 \times 10^{-30} \frac{1}{\omega^2} \alpha^2 \lambda_E^2 \tag{40}$$

where λ_E is normalized to $\frac{c}{\Omega}$. In physical units, Eq. (40) is written in the form

$$F_{lost} = 6.1 \times \frac{1}{\omega^2} \alpha^2 (\frac{c}{\Omega})^2 \lambda_E^2 \tag{41}$$

Now, λ_E is measured in sec while ω and Ω in Hz. However, since $\nu = 1$ we have $\omega = \Omega$. Therefore, Eq. (41), in terms of the proper distance r_E between the source of the gravitational waves and the Earth, is finally written in the form

$$F_{lost} = 6.1 \times 10^{-30} \frac{1}{\omega^4} \alpha^2 r_E^2 \tag{42}$$

where r_E is measured in cm.

The total gravitational energy flux expected in the neighbourhood of the Earth, if we do not take into account any interaction with the interstellar matter (Misner et al 1973, Thorne 1987), is

$$F_{expected} = \frac{c^3}{16\pi G} \omega^2 \alpha^2 = 8.1 \times 10^{36} \omega^2 \alpha^2$$
 (43)

Dividing by parts Eqs. (42) and (43) we obtain

$$\frac{F_{lost}}{F_{expected}} = 0.753 \times 10^{-66} \left(\frac{Hz}{\omega}\right)^6 \left(\frac{r_E}{cm}\right)^2 \tag{44}$$

Eq. (44) gives the percentage of the flux lost due to the resonant interaction between a gravitational wave and a distribution of charged particles, at a proper distance r_E from the source, over the theoretically expected gravitational energy flux, at the same distance, in the absence of resonant damping mechanisms.

It would be interesting to give some numerical examples of the flux damping, for realistic sources of gravitational waves in our galaxy. As those we choose the gravitational systems of bright NS/NS binaries (Thorne 1995). The gravitational waves generated by these sources have very low frequencies, $f = 10^{-4} \ Hz$. Since $\nu = 1$ we have $\Omega = 0.63 \times 10^{-3} \ Hz$ for protons, which is true for a magnetic field of strength $B \simeq 0.5 \times 10^{-6}$ G, a quite reasonable value for the interstellar space (Hillas 1984).

For gravitational waves generated from a binary NS/NS source at a distance $r_E = 10$ Kpc from Earth (the center of our Galaxy), Eq. (44) gives

$$\frac{F_{lost}}{F_{expected}} = 0.0116 = 1.16\%$$

while, for the same source at a distance of $r_E = 30$ Kpc (the other edge of our Galaxy), Eq. (44) gives

$$\frac{F_{lost}}{F_{expected}} = 0.1044 = 10.44\%$$

5 Discussion and conclusions

In the present paper we discuss the resonant interaction of a sinusoidal plane polarized gravitational wave with the interstellar matter, in the presence of a uniform and static in time magnetic field. This sort of interaction arises in the parallel and quasiparallel directions of propagation of the gravitational wave with respect to the dynamic lines of the magnetic field. In any of these cases of propagation and under certain resonant conditions, the trigonometric term in the corresponding Hamiltonian function becomes stationary. The dynamical system is therefore trapped in a resonance (Kleidis et al 1995).

Under an appropriate transformation of the affine parameter, which guarantess that the particles will continue to move along geodesics, we may show that this non-linear interaction problem corresponds to a parametric resonance. In this case the external action (the gravitational wave) amounts to a time variation of the frequency parameter of the initial system (charged particles in a magnetic field), so that in the overall system the equilibrium at rest $(x^1 = 0)$ is unstable. Any deviation from this state, however small, is sufficient to lead to a rapidly increasing displacement along the perpendicular direction (Landau and Lifshitz 1976). Further analysis shows that, in any of the above propagation cases, the overall result would be an exponential increase of the perpendicular energy of the particles involved in this interaction, as a function of the affine parameter and, hence, of their proper-time as well.

However, in this case, except from the primary resonance, there are also many secondary resonances, each one associated to a non-zero resonant width (Landau and Lifshitz 1976). This width is given in terms of the dimensionless amplitude of the gravitational wave, α , and the dimensionless ratio, $\beta \simeq \varrho/\omega^2$, of the energy density of the interstellar plasma over the energy density of the magnetic field (since $\omega^2 \sim E_{magn}$). We may also calculate the total measure of the resonant widths, which is related to the total probability for our dynamical system to be trapped in a resonance. This is a positive quantity, directly proportional to the dimensionless amplitude of the gravitational wave.

Numerical results, in the quasi-parallel case, indicate that an overall increase in the perpendicular energy, I_1 , is also observed in any of the secondary resonances. In fact, in some cases this is true even when the value of I_0 , in terms of which we express each resonant condition, lies outside of the corresponding resonant width. This indicates that the efficiency of the resonant interaction mechanism between charged particles and a gravitational wave is considerably larger than the one obtained in Papers I and II for the following reason.

As a gravitational wave propagates into a non-empty space, it encounters interstellar clouds with different characteristic temperatures, densities and magnetic fields. To each cloud corresponds a specific set of β , ν and ϑ . One expects that, for some of these sets, one of the many resonant conditions will be satisfied, even approximately, leading to a parametric resonant acceleration of the charged particles.

We may give an estimate of the gravitational energy flow lost, due to parametric resonant interaction of the gravitational wave with the interstellar matter, with respect to the flow expected at the Earth in the absence of resonant damping mechanisms. The numerical applications indicate that, for a binary source at a distance of 30 Kpc, the total loss may be up to 10% of the flow theoretically expected. Of course the calculations are somewhat rough and the results refer to the idealized situation in which the wave propagates always parallel to the direction of the magnetic field. However, they indicate that the resonant interaction of a gravitational wave with the interstellar medium could lead to a reevaluation of the today expected values of the wave's amplitude in the neighbourhood of the Earth.

Finally, the analytic results obtained in the present paper generalize the corre-

sponding ones of Papers I and II, since no averaging technique has been used. We note that the main result of the averaging procedure was to smooth out an additional secularily increasing term and the trigonometric modulation, which arise in the exact expression for the perpendicular energy $I_1(\lambda)$.

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